

Spinning-particle model for the Dirac equation and the relativistic *Zitterbewegung*

A. A. Deriglazov*

Depto. de Matemática, ICE, Universidade Federal de Juiz de Fora, MG, Brazil

We construct the relativistic particle model without Grassmann variables which meets the following requirements. A) Canonical quantization of the model implies the Dirac equation. B) The variable which experiences *Zitterbewegung*, represents a gauge non-invariant variable in our model. Hence our particle does not experience the undesirable *Zitterbewegung*. C) In the non-relativistic limit spin is described by three-vector, as it could be expected.

I. INTRODUCTION

The Dirac spinor Ψ can be used to construct the four-dimensional current vector, $\bar{\Psi}\Gamma^\mu\Psi$, which preserves for solutions to the Dirac equation, $\partial_\mu(\bar{\Psi}\Gamma^\mu\Psi) = 0$. As a consequence, the integral over space of its null-component, $\int d^3x \Psi^\dagger\Psi$, does not depend on time. Hence the quantity $\Psi^\dagger\Psi \geq 0$ admits the probabilistic interpretation, and we expect that one-particle sector of the Dirac equation can be described in the framework of relativistic quantum mechanics. Although the true understanding of spin is achieved in the framework of quantum electrodynamics, a lot of efforts has been spent in attempts to construct semiclassical description of relativistic spin on the base of mechanical models [1-12]. However, the relativistic quantum mechanics applied to the one-particle solutions predicts some controversial effects, like the *Zitterbewegung* of the center-of-charge position operator [1, 2] and the Klein's paradox.

Analyzing the applicability of quantum-mechanical treatment to the free Dirac equation, Schrödinger noticed [1] that the center-of-charge position operator in the Heisenberg picture experiences rapid oscillations called *Zitterbewegung*. If we take the state vector with positive and negative energy components, the expectation value of the operator has a similar behavior. It is often assumed that *Zitterbewegung* represents the physically observable motion of a real particle [18, 19]. The analogous systems that are described by a Dirac-type equation and simulate *Zitterbewegung* are under intensive study in different physical set-ups, including graphene, trapped ions, photonic lattices and ultracold atoms (see [22] and the references therein). In this work we show that the status of this phenomenon in relativistic quantum mechanics is not as clear as is commonly believed.

Besides the center of charge, x , in the Dirac theory we can construct the center-of-mass (Pryce-Newton-Wigner) [3, 4] operator \tilde{x} in such a way that the conjugated momentum of x turns out to be the mechanical momentum for \tilde{x} . So the Dirac particle looks like a kind of composed system (this picture has been used by Schrödinger [1] to

identify spin with inner angular momentum of the system). It further complicates the semiclassical analysis, as the Dirac equation gives no evidence as regards which of these two operators should be identified with the position of the particle.

To understand the controversial properties of the one-particle Dirac equation, it would be desirable to have at our disposal the semiclassical particle model which leads to the Dirac equation in the course of canonical quantization. It implies, in particular, that Γ -matrices should be produced through quantization of some set of classical variables. While the problem has a long history (see [3-12] and the references therein), there appears to be no wholly satisfactory solution to date. The main difficulty consists of the proper choice of the basic classical variables for construction of the spin space. If we start from some classical-mechanics action functional, the phase-space variables, say ω^a , π_b , necessarily obey the Poisson bracket algebra $\{\omega^a, \pi_b\} = \delta^a_b$. The number of variables and the algebra are different from the number of spin operators and their commutators (for instance, for non-relativistic spin they are $[\hat{S}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k$). To improve this, we need to impose constraints as well as to pass from the initial to some composed variables. This implies the use of the Dirac machinery for constrained theories [13-16]. Although the non-relativistic spin can be described along these lines [17], it seems to be surprisingly difficult¹ to construct, in a systematic way, a consistent model for the relativistic spin.

In this letter we propose the relativistic particle model without Grassmann (anticommutative) variables which leads to the Dirac equation. We apply our model to analysis of the relativistic *Zitterbewegung*, presenting a simple semiclassical argument on the non-physical character of this phenomenon, which in our model represents the dynamics of an unobservable variable.

*Electronic address: alexei.deriglazov@ufjf.edu.br; On leave of absence from Dep. Math. Phys., Tomsk Polytechnical University, Tomsk, Russia.

¹ The Berezin-Marinov model [9] is based on the Grassmann variables and leads to the Dirac equation. The problem here is that the Grassmann classical mechanics represents a rather formal mathematical construction. It leads to certain difficulties [9, 14] in attempts to use it for description of the spin effects on the semiclassical level, before the quantization. The Hanson-Regge theory [8] does not produce Γ -matrices. The Barut-Zanghi model [10] does not imply the Dirac equation.

II. MODEL-INDEPENDENT CONSTRUCTION OF THE RELATIVISTIC SPIN SURFACE

To construct the relativistic spin surface, we start from the Dirac equation

$$\left[\Gamma^\mu (\hat{p}_\mu + \frac{e}{c} A_\mu) + mc \right] \Psi = 0. \quad (1)$$

Applying the operator $\Gamma^\mu (\hat{p}_\mu + \frac{e}{c} A_\mu) - mc$, this implies the Klein-Gordon equation with non-minimal interaction

$$\left[(\hat{p}^\mu + \frac{e}{c} A^\mu)^2 + \frac{e\hbar}{2c} F_{\mu\nu} \Gamma^{\mu\nu} + m^2 c^2 \right] \Psi = 0, \quad (2)$$

where

$$\Gamma^{\mu\nu} \equiv \frac{i}{2} (\Gamma^\mu \Gamma^\nu - \Gamma^\nu \Gamma^\mu), \quad (3)$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We use the representation with hermitean Γ^0 and antihermitean Γ^i

$$\Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (4)$$

then $[\Gamma^\mu, \Gamma^\nu]_+ = -2\eta^{\mu\nu}$, $\eta^{\mu\nu} = (-+++)$, and $\Gamma^0 \Gamma^i$, Γ^0 are the Dirac matrices α^i , β [2]. We take the classical counterparts of the operators \hat{x}^μ and $\hat{p}_\mu = -i\hbar \partial_\mu$ in the standard way, which are x^μ , p^ν , with the Poisson brackets $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$.

Let us discuss the classical variables that could produce the Γ -matrices. The relativistic equation for the spin precession is usually obtained including the three-dimensional spin vector S^i either into the Frenkel tensor $\Phi^{\mu\nu}$, $\Phi^{\mu\nu} u_\nu = 0$, or into the Bargmann-Michel-Telegdi four-vector² S^μ , $S^\mu u_\mu = 0$, where u_ν represents four-velocity of the particle. However, the semiclassical models based on these schemes do not lead to a reasonable quantum theory, as they do not produce the Dirac equation through canonical quantization. We now present arguments as to how this can be achieved in the formulation that implies inclusion of S^i into the $SO(2,3)$ angular momentum tensor J^{AB} of five-dimensional space, $A = (\mu, 5) = (0, 1, 2, 3, 5)$, with the metric $\eta^{AB} = (-+++-)$.

First, we analyze commutators of the Γ -matrices. The commutators do not form closed Lie algebra, but produce $SO(1,3)$ -Lorentz generators (3). The set Γ^μ , $\Gamma^{\mu\nu}$ forms closed algebra

$$\begin{aligned} [\Gamma^\mu, \Gamma^\nu] &= -2i\Gamma^{\mu\nu}, & [\Gamma^{\mu\nu}, \Gamma^\alpha] &= 2i(\eta^{\mu\alpha}\Gamma^\nu - \eta^{\nu\alpha}\Gamma^\mu), \\ [\Gamma^{\mu\nu}, \Gamma^{\alpha\beta}] &= 2i(\eta^{\mu\alpha}\Gamma^{\nu\beta} - \eta^{\mu\beta}\Gamma^{\nu\alpha} - \eta^{\nu\alpha}\Gamma^{\mu\beta} + \eta^{\nu\beta}\Gamma^{\mu\alpha}). \end{aligned} \quad (5)$$

The algebra can be identified with the five-dimensional Lorentz algebra $SO(2,3)$ with generators \hat{J}^{AB}

$$[\hat{J}^{AB}, \hat{J}^{CD}] = 2i(\eta^{AC}\hat{J}^{BD} - \eta^{AD}\hat{J}^{BC} -$$

$$\eta^{BC}\hat{J}^{AD} + \eta^{BD}\hat{J}^{AC}), \quad (6)$$

assuming $\Gamma^\mu \equiv \hat{J}^{5\mu}$, $\Gamma^{\mu\nu} \equiv \hat{J}^{\mu\nu}$.

To reach the algebra starting from a classical-mechanics model, we introduce ten-dimensional "phase" space of the spin degrees of freedom, ω^A , π^B , equipped with the Poisson bracket $\{\omega^A, \pi^B\} = \eta^{AB}$. Consider the inner angular momentum

$$J^{AB} \equiv 2(\omega^A \pi^B - \omega^B \pi^A). \quad (7)$$

Poisson brackets of these quantities form the algebra

$$\{J^{AB}, J^{CD}\}_{PB} = 2(\eta^{AC} J^{BD} - \eta^{AD} J^{BC} - \eta^{BC} J^{AD} + \eta^{BD} J^{AC}). \quad (8)$$

Comparing (8) with (6) we conclude that the operators Γ^μ , $\Gamma^{\mu\nu}$ could be obtained by quantization of J^{AB} .

Since J^{AB} are the variables which we are interested in, we try to take them as coordinates of the space ω^A , π^B . The Jacobian of the transformation $(\omega^A, \pi^B) \rightarrow J^{AB}$ has rank equal seven³. So, only seven among ten functions $J^{AB}(\omega, \pi)$, $A < B$, are independent quantities. They can be separated as follows. By construction, the quantities (7) obey the identity $\epsilon^{\mu\nu\alpha\beta} J^5{}_\nu J_{\alpha\beta} = 0$, this can be solved as

$$J^{ij} = (J^{50})^{-1} (J^{5i} J^{0j} - J^{5j} J^{0i}). \quad (9)$$

Hence we can take $J^{5\mu}$, J^{0i} as the independent variables. We could complete the set up to a base of the phase space (ω^A, π^B) adding three more coordinates, for instance ω^3 , ω^5 , π^5 . Quantizing the complete set we obtain, besides the desired operators $\hat{J}^{5\mu}$, \hat{J}^{0i} , some extra operators $\hat{\omega}^3$, $\hat{\omega}^5$, $\hat{\pi}^5$. They are not present in the Dirac theory, and are not necessary for description of spin. So we need to reduce the dimension of our space from ten to seven imposing three constraints. There is one important restriction on the choice of constraints. Canonical quantization of a system with constraints implies replacement the Poisson by the Dirac bracket, the latter is constructed with help of the constraints. We need $SO(2,3)$ -invariant constraints T_a , $\{T_a, J^{AB}\}_{PB} = 0$, otherwise the Dirac-brackets algebra will not coincide with those of the Poisson, (8).

The only quadratic $SO(2,3)$ -invariants which can be constructed from ω^A , π^B are $\omega^A \omega_A$, $\omega^A \pi_A$ and $\pi^A \pi_A$. So we restrict our model to live on the surface defined by the equations

$$T_3 \equiv \pi^A \pi_A + a_3 = 0; \quad (10)$$

$$T_4 \equiv \omega^A \omega_A + a_4 = 0, \quad T_5 \equiv \omega^A \pi_A = 0, \quad (11)$$

where a_3 , a_4 are some numbers.

² The conditions $\Phi^{\mu\nu} u_\nu = 0$ and $S^\mu u_\mu = 0$ guarantee that in the rest frame only three components of these quantities survive, which implies the right non-relativistic limit.

³ The rank has been computed using the program: Wolfram Mathematica 8.

The matrix $\frac{\partial(J^{5\mu}, J^{0i}, T_4, T_5, \omega^5)}{\partial(\omega^A, \pi^B)}$ has rank equal ten. So the quantities

$$J^{5\mu}, J^{0i}, T_4, T_5, \omega^5, \quad (12)$$

can be taken as coordinates of the space (ω^A, π^B) . The equation $J^{AB} = 2(\omega^A \pi^B - \omega^B \pi^A)$ implies the identity

$$J^{AB} J_{AB} = 8[(\omega^A)^2 (\pi^B)^2 - (\omega^A \pi_A)^2] = 8[(T_4 - a_4)(T_3 - a_3) - (T_5)^2], \quad (13)$$

then the constraint T_3 can be written in the coordinates (12) as follows:

$$T_3 = \frac{(J^{AB})^2 + 8(T_5)^2}{8(T_4 - a_4)} + a_3, \quad (14)$$

where J^{ij} are given by Eq. (9). Note that T_3 does not depend on ω^5 . On the hyperplane $T_4 = T_5 = 0$ it reduces to

$$-8a_4 T_3 = (J^{AB})^2 - 8a_3 a_4 = 0. \quad (15)$$

Eq. (15) states that the value of $SO(2,3)$ -Casimir operator⁴ $(J^{AB})^2$ is equal to $8a_3 a_4$.

In the dynamical model constructed below, the equation $T_3 = 0$ appears as the first-class constraint. It implies that we deal with a theory with local symmetry, with the constraint being the generator of the symmetry [20]. The coordinate ω^5 is not inert under the symmetry, $\delta\omega^5 \sim \{T_3, \omega^5\} \neq 0$. Hence ω^5 is gauge non-invariant variable.

Summing up, we have restricted dynamics of spin on the surface (10), (11). If (12) are taken as coordinates of the phase space, the surface is the hyperplane $T_4 = T_5 = 0$ with the coordinates $J^{5\mu}, J^{0i}, \omega^5$ subject to the condition (15). Since ω^5 is gauge non-invariant coordinate, we can discard it. It implies that we can quantize $J^{5\mu}, J^{0i}$ instead of the initial variables ω^A, π^B .

Following the canonical quantization paradigm, the variables must be replaced by Hermitian operators⁵ with commutators resembling the Poisson bracket

$$[\ , \] = i\hbar \{ \ , \ }|_{J \rightarrow \hat{J}}. \quad (16)$$

Similarly to the case of Γ -matrices, brackets of the variables $J^{5\mu}, J^{0i}$ do not form closed Lie algebra. The non closed brackets are

$$\{J^{5i}, J^{5j}\} = \{J^{0i}, J^{0j}\} = -2J^{ij}, \quad (17)$$

where J^{ij} is given by Eq. (9). Adding them to the initial variables, we obtain the set $J^{AB} = (J^{5\mu}, J^{0i}, J^{ij})$ which obeys the desired algebra (8).

According to Eqs. (6), (8) the quantization is achieved replacing the classical variables $J^{5\mu}, J^{\mu\nu}$ on Γ -matrices⁶. We assume that ω^A has a dimension of length, then J^{AB} has the dimension of the Planck's constant. Hence the quantization rule is

$$J^{5\mu} \rightarrow \hbar\Gamma^\mu, \quad J^{\mu\nu} \rightarrow \hbar\Gamma^{\mu\nu}. \quad (18)$$

This implies that the Dirac equation (1) can be produced by the constraint (since the *Zitterbewegung* is a property of the free Dirac equation, we take temporarily $A_\mu = 0$)

$$T_2 \equiv p_\mu J^{5\mu} + m\hbar c = 0. \quad (19)$$

In quantum theory, the Dirac equation implies the Klein-Gordon one. In contrast, in the classical theory the constraint (19) does not imply the mass-shell condition

$$T_1 \equiv p^2 + m^2 c^2 = 0. \quad (20)$$

To improve this, we are forced to look for a classical model that produces this equation as an independent constraint. The model without the constraint (20) has been considered in [21]. It shows the same undesirable properties as those of Dirac equation in the classical limit.

III. SPINNING-PARTICLE ACTION, CANONICAL QUANTIZATION AND THE DIRAC EQUATION

According to the previous section, to describe the relativistic spin we need a theory that implies the Dirac constraints (10), (11), (19), (20).

We recall that the Hamiltonian action for a system with the phase-space variables Q^α, P_α reads $S = P_\alpha \dot{Q}^\alpha - H$, $H = H_0 + \lambda_a \Phi_a$, where H_0 is the canonical Hamiltonian and λ_a are the Lagrangian multipliers for the primary constraints Φ_a . For the present case, we propose to consider the total Hamiltonian of the form $H = \frac{1}{2}e_a T_a + \lambda_{ea} \pi_{ea}$, where π_{ea} are conjugate momenta for the auxiliary variables e_a , the latter are associated with the constraints T_a , $a = 1, 2, 3, 4$. So, let us consider the following $d = 4$ Poincaré invariant action

$$S = \int d\tau p_\mu \dot{x}^\mu + \pi_A \dot{\omega}^A - \left[\frac{e_1}{2}(p^2 + m^2 c^2) + \frac{e_2}{2}(p_\mu J^{5\mu} + m\hbar c) + \frac{e_3}{2}T_3 + \frac{e_4}{2}T_4 + \pi_{ea}(\lambda_{ea} - \dot{e}_a) \right], \quad (21)$$

If we omit the spin-space coordinates, $\omega^A = \pi_A = 0$, Eq. (21) reduces to the well known action of a spinless relativistic particle

$$S_0 = \int d\tau p_\mu \dot{x}^\mu - \frac{1}{2}e(p^2 + m^2 c^2) + \pi_e(\dot{e} - \lambda_e). \quad (22)$$

⁴ In quantum theory, for the operators (6), (5) we have: $\hat{J}^{AB} \hat{J}_{AB} = 20\hbar^2$.

⁵ The matrices $\Gamma^\mu, \Gamma^{\mu\nu}$ are Hermitian operators with respect to the scalar product $(\Psi_1, \Psi_2) = \Psi_1^\dagger \Gamma^0 \Psi_2$.

⁶ Replacing (9) by an operator $\hat{J}^{ij}(\Gamma^\mu, \Gamma^{0i})$ we arrange the operators Γ in such a way, that $\hat{J}^{ij}(\Gamma) = \Gamma^{ij}$.

Variation of the action (21) with respect to e_a leads to the constraints $T_a = 0$. Preservation in time of the constraint T_4 , $\dot{T}_4 = 0$, implies $T_5 = 0$, that is we reproduce all the desired constraints (10), (11), (19), (20). The constraints T_1, T_2 have vanishing Poisson brackets with all the constraints. The remaining constraints obey the Poisson-bracket algebra

$$\begin{aligned} \{T_3, T_4\} &= -4T_5, & \{T_3, T_5\} &= -2T_3 + 2a_3, \\ \{T_4, T_5\} &= 2T_4 - 2a_4. \end{aligned} \quad (23)$$

If we take the combination

$$\tilde{T}_3 \equiv T_3 + \frac{a_3}{a_4} T_4, \quad (24)$$

the algebra acquires the form

$$\begin{aligned} \{\tilde{T}_3, T_4\} &= -4T_5, & \{\tilde{T}_3, T_5\} &= -2T_3 + 2\frac{a_3}{a_4} T_4, \\ \{T_4, T_5\} &= 2T_4 - 2a_4. \end{aligned} \quad (25)$$

The only bracket which does not vanish on the constraint surface is $\{T_4, T_5\}$. According the Dirac terminology [13-15], we have the first-class constraints (24), (19), (20), and the second-class pair (11). The presence of the first-class constraints indicates that we are dealing with a theory invariant under a three-parameter group of local (gauge) symmetries, which will be found below.

The constraints (24), (11) can be taken into account by transition from the Poisson to the Dirac bracket, and after that they can be omitted from consideration [13-15]. Since the Dirac brackets are constructed with use of $SO(2,3)$ -invariants, the Dirac brackets of the quantities J^{AB} coincide with the Poisson one (8). Hence we quantize the model according Eq. (18). The operator produced by the first-class constraint (19) is imposed on the state vector. This gives the Dirac equation. In the result, canonical quantization of the model leads to the desired quantum picture.

The Lagrangian formulation can be restored from Eq. (21). To achieve this, we note that the conjugate momenta enter into the Hamiltonian in the form $\frac{1}{2}P_\alpha G^{\alpha\beta} P_\beta$, where $P_\alpha \equiv (p_\mu, \pi_\nu, \pi^5)$ and the "metric" $G^{\alpha\beta}(\omega^\mu, \omega^\nu, e_1, e_2, e_3)$ is given by 9×9 -matrix

$$\left(\begin{array}{ccc|ccc|c} & e_1\eta & & e_2\omega^5\eta & & & -e_2\omega^0 \\ & - & - & - & - & - & \dots \\ & & & & & & -e_2\omega^3 \\ & e_2\omega^5\eta & & e_3\eta & & & 0 \\ & - & - & - & - & - & \dots \\ & & & & & & 0 \\ -e_2\omega^0 & \dots & -e_2\omega^3 & 0 & \dots & 0 & -e_3 \end{array} \right)$$

Here η is the Minkowski metric. To find the Lagrangian, we solve the Hamiltonian equations for the position variables $Q^\alpha \equiv (x^\mu, \omega^\nu, \omega^5)$, $\dot{Q}^\alpha = G^{\alpha\beta} P_\beta$, with respect to P_α . It gives $P_\alpha = G_{\alpha\beta} \dot{Q}^\beta$, where $G_{\alpha\beta}$ is the inverse metric. We substitute these P_α back into Eq. (21), obtaining

the Lagrangian

$$L = \frac{1}{2} G_{\alpha\beta} \dot{Q}^\alpha \dot{Q}^\beta - \frac{1}{2} e_4 \omega^A \omega_A - \frac{1}{2} e_b a_b. \quad (26)$$

It has been denoted $(a_1, a_2, a_3, a_4) = (m^2 c^2, m c \hbar, a_3, a_4)$. The kinetic term looks like those of a free particle moving on the curved nine-dimensional space with the metric $G_{\alpha\beta}$.

We introduce the abbreviation

$$Dx^\mu \equiv \dot{x}^\mu - \frac{e_2}{e_3} J_L^{5\mu}, \quad (27)$$

where $J_L^{5\mu} \equiv \omega^5 \dot{\omega}^\mu - \omega^\mu \dot{\omega}^5$ is the configuration-space counterpart of the angular momentum $J^{5\mu}$. Then manifest form of the Lagrangian (26) is

$$\begin{aligned} 2L &= \frac{e_3}{B} \left[(Dx^\mu)^2 - \frac{e_2^2}{A} (Dx^\mu \omega_\mu)^2 \right] + \\ &\frac{1}{e_3} \dot{\omega}^A \dot{\omega}_A - e_4 \omega^A \omega_A - e_b a_b. \end{aligned} \quad (28)$$

It has been denoted $B = e_1 e_3 - e_2^2 (\omega^5)^2$, $A = B + e_2^2 (\omega^\mu)^2$. We have verified that hamiltonization of the Lagrangian leads back to the Hamiltonian action (21).

IV. CLASSICAL EQUATIONS OF MOTION AND THE LACK OF ZITTERBEWEGUNG

In this section we show that the status of *Zitterbewegung* phenomenon in relativistic quantum mechanics is not as clear as is commonly believed.

Besides the constraints discussed above, the action (21) implies the following equations (we use the notation $(p\omega) = p^\mu \omega_\mu$)

$$\begin{aligned} \dot{e}_a &= \lambda_{ea}, \quad \pi_{ea} = 0, \quad a = 1, 2, 3; \\ e_4 &= \frac{a_3}{a_4} e_3, \quad \pi_{e_4} = 0, \quad \lambda_{e_4} = \frac{a_3}{a_4} \lambda_{e_3}; \end{aligned} \quad (29)$$

$$\begin{aligned} \dot{x}^\mu &= e_1 p^\mu + \frac{1}{2} e_2 J^{5\mu}, \quad \dot{p}^\mu = 0; \\ \dot{\omega}^\mu &= e_3 \pi^\mu + e_2 \omega^5 p^\mu, \quad \dot{\pi}^\mu = e_2 \pi^5 p^\mu - \frac{a_3}{a_4} e_3 \omega^\mu; \\ \dot{\omega}^5 &= e_3 \pi^5 + e_2 (p\omega), \quad \dot{\pi}^5 = e_2 (p\pi) - \frac{a_3}{a_4} e_3 \omega^5. \end{aligned} \quad (30)$$

As a consequence, we obtain (assuming the gauge conditions $e_a = \text{const}$) $\ddot{x}^\mu = -\frac{1}{2} e_2^2 J^{\mu\nu} p_\nu$, that is x^μ -variable is under *Zitterbewegung* in non-interacting theory.

Since we deal with a constrained theory, our first task is to specify the physical-sector variables. We note that Eqs. (29), (30) do not determine the Lagrangian multipliers $\lambda_a(\tau)$, $a = 1, 2, 3$. Eqs. (29) then imply that e_a , $a = 1, 2, 3$, remain an arbitrary functions as well. The arbitrary functions enter into solutions to equations of motion for the variables x^μ, ω^A, π^A . Hence, except p^μ , all the variables has ambiguous dynamics. According to

the general theory [13-15], variables with ambiguous dynamics do not represent the observable quantities. In particular, the coordinate x^μ , which corresponds to the center-of-charge position operator of the Dirac equation, and experiences *Zitterbewegung*, is an unobservable quantity in our model.

Let us compute the total number of physical degrees of freedom. Omitting the auxiliary variables and the corresponding constraints, we have 18 phase-space variables $x^\mu, p_\mu, \omega^A, \pi_A$ subject to the constraints (10), (11), (19), (20). Taking into account that each second-class constraint rules out one variable, whereas each first-class constraint rules out two variables, the number of physical degrees of freedom is $18 - (2 + 2 \times 3) = 10$. To construct the unambiguous position variable, we note that the quantity

$$\tilde{x}^\mu = x^\mu + \frac{1}{2p^2} J^{\mu\nu} p_\nu, \quad (31)$$

obeys $\dot{\tilde{x}}^\mu = \tilde{e} p^\mu$, where $\tilde{e} \equiv e_1 + \frac{\hbar e_2}{2mc}$. Besides, we know $\dot{p}_\mu = 0$. Since these equations resemble those for a spinless relativistic particle, the remaining ambiguity due to \tilde{e} has the well-known interpretation, being related with reparametrization invariance of the theory. In accordance with this, we assume that $\tilde{x}^\mu(\tau)$ represent the parametric equations of the trajectory $\tilde{x}^i(t)$. Using the identity $\frac{dA(t)}{dt} = c \frac{\dot{A}(\tau)}{\dot{x}^0(\tau)}$, we conclude that the reparametrization-invariant variable $\tilde{x}^i(t)$ has deterministic evolution: $\frac{d\tilde{x}^i}{dt} = c \frac{\dot{\tilde{x}}^i}{\dot{x}^0} = \frac{cp^i}{p^0}$. In the absence of interaction, it moves along a straight line. We also notice that \tilde{x}^μ represents the center-of-mass (Pryce-Newton-Wigner) coordinate [3, 4], and p_μ represents its mechanical momentum. Hence the mass-shell condition (20) guarantees that the \tilde{x}^i -particle cannot exceed the speed of light.

As the classical four-dimensional spin vector, we take the Pauli-Lubanski vector $S^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta}$. It has no precession in the free theory, $\dot{S}^\mu = 0$. In the rest frame $p^\mu = (mc, 0, 0, 0)$, it reduces to the three-dimensional rotation generator, $S^0 = 0$, $S^i = \frac{1}{2} m c \epsilon^{ijk} S_{jk}$, as is expected in the non-relativistic limit. Hence the ten variables $x^i(t), p^i(t), S^\mu(t)$ can be taken as the physical variables.

We have specified the physical sector from analysis of equations of motion. The more traditional way to do this consists of analysis of local symmetries of the formulation. The action (21) is invariant under a three-parameter group of local symmetries. One of them is the reparametrization symmetry which we take in the form

$$\delta_\alpha Z = \alpha \{Z, H\}, \quad \delta_\alpha e_a = (\alpha e_a). \quad (32)$$

Here $Z = (x^\mu, p^\mu, \omega^A, \pi_A)$. This form of reparametrization symmetry can be obtained as follows. It is sufficient to consider the spinless particle action (22). The standard form of reparametrization symmetry reads $\delta x^\mu = \alpha \dot{x}^\mu$, $\delta p^\mu = \alpha \dot{p}^\mu$, $\delta e = (\alpha e)$ (we have omitted the transformations $\delta \lambda_e = (\delta e)$ and $\delta \pi_e = 0$ which are not

relevant for our discussion). We rewrite the symmetry in equivalent form, without derivatives acting on x and p . Every Hamiltonian action has trivial symmetries [20], in this case they are $\delta x^\mu = \epsilon(\dot{x}^\mu - \{x^\mu, H\})$, $\delta p^\mu = \epsilon(\dot{p}^\mu - \{p^\mu, H\})$, where H stands for the complete Hamiltonian. Taking the combination $\delta_\alpha + \delta_\epsilon$ with $\epsilon = -\alpha$, the reparametrization invariance acquires the form $\delta x^\mu = \alpha \{x^\mu, H\}$, $\delta p^\mu = \alpha \{p^\mu, H\}$, $\delta e = (\alpha e)$.

For our case (21), the manifest form of the reparametrization symmetry is

$$\begin{aligned} \delta x^\mu &= \alpha(e_1 p^\mu + \frac{1}{2} e_2 J^{5\mu}), & \delta p^\mu &= 0, & \delta e_a &= (\alpha e_a), \\ \delta \omega^\mu &= \alpha(e_3 \pi^\mu + e_2 \omega^5 p^\mu), & \delta \pi^\mu &= \alpha(e_2 \pi^5 p^\mu - \frac{a_3}{a_4} e_3 \omega^\mu), \\ \delta \omega^5 &= \alpha(e_3 \pi^5 + e_2(p\omega)), & \delta \pi^5 &= \alpha(e_2(p\pi) - \frac{a_3}{a_4} e_3 \omega^5). \end{aligned}$$

Besides, there are two more symmetries with the local parameters $\beta(\tau), \gamma(\tau)$

$$\delta_\beta x^\mu = \beta p^\mu, \quad \delta_\beta e_1 = \dot{\beta}; \quad (33)$$

$$\delta_\gamma \omega^A = \gamma \pi^A, \quad \delta_\gamma e_3 = \dot{\gamma}. \quad (34)$$

All the initial variables, except p_μ , are not gauge invariant. The center of mass \tilde{x}^μ turns out to be invariant with respect to two symmetries: $\delta_\gamma \tilde{x}^\mu = 0$, $\delta_{\tilde{\beta}} \tilde{x}^\mu = 0$, where $\delta_{\tilde{\beta}} = \delta_\beta + \delta_\alpha$, $\alpha = -\frac{2p^2 \beta}{2p^2 e_1 + e_2(pJ^5)}$. As expected, \tilde{x}^μ is affected only by reparametrizations: $\delta_\alpha \tilde{x}^\mu = \alpha(e_1 + \frac{e_2(pJ^5)}{2p^2}) p^\mu$. The spin vector S^μ is invariant under all the transformations. So we have confirmed our previous result: \tilde{x}^μ, p_μ and S^μ can be taken as the physical-sector variables.

We finish with a preliminary comment on interaction with an external electromagnetic field. The classical constraints that produce Eqs. (1), (2) are $\tilde{T}_2 \equiv (p_\mu + \frac{e}{c} A_\mu) J^{5\mu} + mc\hbar = 0$, $\tilde{T}_1 \equiv (p^\mu + \frac{e}{c} A^\mu)^2 + \frac{e}{2c} F_{\mu\nu} J^{\mu\nu} + m^2 c^2 = 0$. Their Poisson bracket reads $\{\tilde{T}_2, \tilde{T}_1\} = -\frac{e}{2c} \partial_\alpha F_{\mu\nu} J^{5\alpha} J^{\mu\nu}$. For the homogeneous electric and magnetic fields, the constraints form the first-class system, $\{\tilde{T}_2, \tilde{T}_1\} = 0$. Hence the interaction does not break the local symmetries presented in our model. In the general case, the breaking is in a sense "soft", i.e., proportional to \hbar^2 . Hence, to construct an interaction with an arbitrary field, one can start the iteration procedure, adding the non-minimal-interaction terms of order \hbar^2 or more to the constraints \tilde{T}_1, \tilde{T}_2 .

V. CONCLUSION

In this work we have constructed a semiclassical model (21), (28) for description of the relativistic spin and showed its consistency both on the classical and on the quantum level. Canonical quantization of the model leads to the Dirac equation. As we could expect for the relativistic spinning particle, the physical sector of the

model is composed by the position variable \tilde{x}^μ (31) and the spin vector S^μ . In the absence of interaction, they obey the free equations $\ddot{\tilde{x}}^\mu = 0$, $\dot{S}^\mu = 0$.

We have presented a simple semiclassical argument that prohibits the relativistic *Zitterbewegung*. Roughly speaking, the argument is as follows. The Dirac equation (1) implies the Klein-Gordon one (2). In contrast, in the classical theory the corresponding constraint (19) does not imply the mass-shell constraint $p^2 + m^2 c^2 = 0$. To obtain a consistent picture, we are forced to construct the semiclassical model that produces both constraints. In turn, the presence of independent constraints implies that we deal with a theory with the local symmetries (32)-(34). Physical quantities are those invariant under

the local symmetries. Our observation is that the classical variable x^μ , that corresponds to the center-of-charge position operator of the Dirac theory, and experiences *Zitterbewegung*, is not invariant. Similarly to the potential A^μ of electromagnetic field, x^μ is an unobservable quantity.

VI. ACKNOWLEDGMENTS

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